

MINIMAL VOLUME k -POINT LATTICE d -SIMPLICES

HAN DUONG

ABSTRACT. We show via triangulations that for $d \geq 3$ there is exactly one class (under unimodular equivalence) of nondegenerate lattice simplices in \mathbb{R}^d with minimal volume and k interior lattice points.

1. INTRODUCTION

A d -polytope P is a polytope of dimension d . If its *vertex set* $\mathcal{V}(P)$ is a subset of \mathbb{Z}^d , then P is a *lattice d -polytope*. If in addition $|\mathcal{V}(P)| = d + 1$, then P is a *lattice d -simplex*. The convex hull of $\mathcal{P} = \{v_1, \dots, v_n\} \subset \mathbb{Z}^d$, denoted by $\text{conv}(\mathcal{P})$, is a lattice polytope with at most n vertices and dimension at most d . This notation will be used loosely; for convenience, we use $\text{conv}(P, v)$ to mean $\text{conv}(\mathcal{V}(P) \cup \{v\})$ when it is clear P is a polytope and v is a point. As used in [23], we say that P is *clean* if $\partial P \cap \mathbb{Z}^d = \mathcal{V}(P)$, where ∂P is the boundary of P . If in addition $\text{int}(P)$, the *interior* of P , contains k lattice points, then P is a *clean k -point lattice polytope*. If $k = 0$, then the polytope is *empty*. We use \mathcal{S}_k^d to denote the collection of clean k -point lattice d -simplices. Unless otherwise stated, all polytopes are taken to be convex d -polytopes.

Reznick proved in [22] and [23] that any lattice tetrahedron with at least one clean face is unimodularly equivalent to some $T_{a,b,n}$, the lattice tetrahedron with vertex set

$$\{ (0, 0, 0), (1, 0, 0), (0, 1, 0), (a, b, n) \}$$

where $(a, b, n) \in \mathbb{Z}^3$ and $0 < a, b < n$. Reznick also classified the set of clean 1-point tetrahedra, up to equivalence under unimodular transformations, using barycentric coordinates. Very recently, Bey, Henk, and Wills proved in [2] that if P is a lattice d -polytope, not necessarily clean, and P has k interior lattice points, then for $d \geq 1$, the volume of P satisfies

$$(1) \quad \text{Vol}(P) \geq \frac{1}{d!}(dk + 1).$$

Moreover, they showed that for $k = 1$, equality holds if and only if P is unimodularly equivalent to the simplex $S_d(1)$, where

$$S_d(k) = \text{conv} \left(e_1, \dots, e_d, -k \sum_{i=1}^d e_i \right)$$

and e_i denotes the i -th unit point. This is not true for $d = 2$ and $k > 2$. We will show that equality holds in (1) for all $k > 0$ if and only if $d \geq 3$ and P is unimodularly equivalent to $S_d(k)$. We first prove that if $T \in \mathcal{S}_k^d$ and $\text{Vol}(T) = \frac{1}{d!}(dk + 1)$, then the interior points lie on a line passing through some vertex of T . We then show that such simplices are unimodularly equivalent to T_{a_1, \dots, a_d} , the d -simplex

whose vertex set consists of the origin, the points e_i ($1 \leq i \leq d-1$), and the point (a_1, \dots, a_d) , where $a_j = dk$ for $1 \leq j < d$ and $a_d = dk + 1$. Finally, we will show that $S_d(k) \in \mathcal{S}_k^d$ and $\text{Vol}(S_d(k)) = \frac{1}{d!}(dk + 1)$.

2. PRELIMINARIES

The following definitions are taken from [1], [7], [17], and [24]. A j -flat is a j -dimensional affine subspace of \mathbb{R}^d . *Points*, *lines*, and *planes* are 0-flats, 1-flats, and 2-flats, respectively. The *affine hull* of a set $\mathcal{P} \subset \mathbb{R}^d$, denoted by $\text{aff}(\mathcal{P})$, is the intersection of all flats containing \mathcal{P} . Equivalently, $\text{aff}(\mathcal{P})$ is the smallest flat containing \mathcal{P} . We say \mathcal{P} is in *general position* if no $j+2$ points of \mathcal{P} lie in a j -flat, where $j < d$. A hyperplane

$$H = \{x \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\}$$

is a $(d-1)$ -flat. If P is a d -polytope, then H is a *supporting hyperplane* of P if P lies entirely on one side of H . A *face* of P is an intersection $P \cap H$, where H is a supporting hyperplane. If we allow degenerate hyperplanes, then P is a face of P corresponding to $H = \mathbb{R}^d$; \emptyset is also a face of P corresponding to a hyperplane that does not meet P . A j -face of P is a j -dimensional face of P . A $(d-1)$ -face is a *facet*, a 1-face is an *edge*, and a 0-face is a *vertex*. One property of d -polytopes is that any j -face of P is contained in at least $d-j$ facets of P .

Lemma 1. [7, Section 3.1] *If $0 \leq i < j \leq d-1$ and if P is a d -polytope, each i -face of P is the intersection of the family of j -faces of P containing it. There are at least $j+1-i$ such faces.*

We generally use capital letters to denote d -polytopes. In particular P and Q are d -polytopes, and S and T are d -simplices. Capital script letters will generally denote sets of d -simplices. In particular, \mathcal{P} is a set of points (0-simplices).

3. TRIANGULATIONS AND REFINEMENTS

Borrowing from [15], we let \mathcal{P} denote a set of n distinct points in \mathbb{R}^d , where $n \geq d+1$ and $d \geq 2$. Assume \mathcal{P} does not lie entirely in a hyperplane. Let $P = \text{conv}(\mathcal{P})$. A *triangulation*, \mathcal{T} , of \mathcal{P} (or of P with the dependence on \mathcal{P} understood) is a set of nondegenerate d -simplices $\{T_i\}$ with the following properties.

- (a) All vertices of each simplex are members of \mathcal{P} .
- (b) The interiors of the simplices are pairwise disjoint.
- (c) Each facet of a simplex is either on the boundary of P , or else is a common facet of exactly two simplices.
- (d) Each simplex contains no points of \mathcal{P} other than its vertices.
- (e) The union of $\{T_i\}$ is \mathcal{P} and the union of T_i is P .

Since each d -simplex has volume at least $\frac{1}{d!}$, one immediate consequence is

$$(2) \quad \text{Vol}(P) \geq \frac{1}{d!} \cdot |\mathcal{T}|.$$

To prove (1), Bey, Henk, and Wills showed P with k interior lattice points can be decomposed into at least $dk+1$ nondegenerate d -subpolytopes. Any d -polytope must contain a d -simplex as a subpolytope, so (2) still holds if \mathcal{T} is replaced by this decomposition. We will present a slight variation of their theorem and its proof by using triangulations.

Definition. Let P be a lattice d -polytope. A *lattice triangulation* \mathcal{T} of P is a triangulation of some set $\mathcal{P} \subset \mathbb{Z}^d$ such that $\mathcal{V}(P) \subseteq \mathcal{P} \subseteq P \cap \mathbb{Z}^d$. Note that if $\mathcal{V}(P) \subseteq \mathcal{P}, \mathcal{P}' \subseteq P \cap \mathbb{Z}^d$ and $\mathcal{P} \neq \mathcal{P}'$, we still have $\text{conv}(\mathcal{P}) = P = \text{conv}(\mathcal{P}')$. On the other hand, the triangulation \mathcal{T} of \mathcal{P} and the triangulation \mathcal{T}' of \mathcal{P}' are necessarily different. Fortunately, we can reconstruct the *vertex set* of \mathcal{T} , denoted by $\mathcal{V}(\mathcal{T})$, by taking the union of all vertices of all $T \in \mathcal{T}$. Thus $\mathcal{V}(\mathcal{T}) = \mathcal{P}$ and $\mathcal{V}(\mathcal{T}') = \mathcal{P}'$.

Definition. Let P be a lattice polytope and \mathcal{T} be a lattice triangulation P . We say \mathcal{T}' is a *refinement* of \mathcal{T} (and write $\mathcal{T} < \mathcal{T}'$) provided \mathcal{T}' is a lattice triangulation of P , $\mathcal{V}(\mathcal{T}) \subsetneq \mathcal{V}(\mathcal{T}')$, and for all $T' \in \mathcal{T}'$ there exists $T \in \mathcal{T}$ such that $T' \subseteq T$. We say that \mathcal{T} is a *full lattice triangulation* if $\mathcal{V}(\mathcal{T}) = P \cap \mathbb{Z}^d$. Otherwise we say \mathcal{T} is a *partial lattice triangulation*.

Naturally, triangulations of $\mathcal{P} \subset \mathbb{R}^d$ partition $\text{conv}(\mathcal{P})$ into simplices. On the other hand, there exist partitions of $\text{conv}(\mathcal{P})$ that satisfy all but condition (d) in the definition of a triangulation.

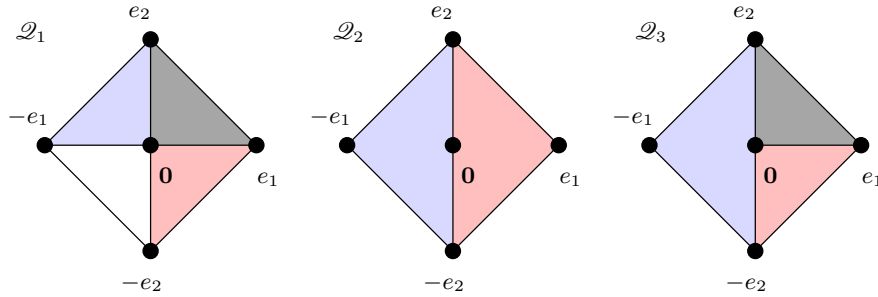


FIGURE 1. Partitions of P into triangles

Example 1. Let $\mathcal{P} = \{e_1, e_2, -e_1, -e_2\} \subset \mathbb{R}^2$ and let $P = \text{conv}(\mathcal{P})$. Note that P is a 1-point lattice polygon, and its interior lattice point is the origin. Three possible partitions \mathcal{Q}_1 , \mathcal{Q}_2 , and \mathcal{Q}_3 of P into triangles are shown in Figure 1. Note that \mathcal{Q}_1 is a triangulation of $\mathcal{P} \cup \{(0, 0)\}$ and a full lattice triangulation of P . However, \mathcal{Q}_1 is not a triangulation of \mathcal{P} . The middle partition \mathcal{Q}_2 is both a triangulation of \mathcal{P} and a partial lattice triangulation of P (viewed as a triangulation of $\mathcal{V}(P) = \mathcal{P}$). The partition \mathcal{Q}_3 on the right is not a triangulation since it fails condition (d).

The following theorem, the proof of which can be found in the appendix of [1], guarantees the existence of a triangulation of the vertex set of a polytope.

Theorem 2. [1, Theorem 3.1] *Every convex polytope P can be triangulated using no new vertices. That is, there exists a triangulation of $\mathcal{V}(P)$.*

Let P be a clean, non-empty lattice d -polytope, and suppose $w \in \text{int}(P) \cap \mathbb{Z}^d$. We construct a *basic lattice triangulation* \mathcal{T}_w of P in the following manner. Theorem 2 guarantees that each facet F of P , as a $(d-1)$ -polytope, has a (lattice) triangulation \mathcal{T}_F of $\mathcal{V}(F)$. Let \mathcal{F} be the set of facets of P and let \mathcal{B} be the set

$$\mathcal{B} = \bigcup_{F \in \mathcal{F}} \mathcal{T}_F$$

of $(d-1)$ -simplices. Finally, let $\mathcal{T}_w = \{ \text{conv}(S, w) : S \in \mathcal{B} \}$. It is easy to check that \mathcal{T}_w is a lattice triangulation of P . Note that

$$(3) \quad |\mathcal{T}_w| = |\mathcal{B}| \geq |\mathcal{F}| \geq d+1.$$

In particular, if $T \in \mathcal{S}_k^d$, where $k \geq 1$, and F_1, \dots, F_{d+1} are the facets of T , then a basic lattice triangulation \mathcal{T}_w is simply the convex hull of the facets of T with an interior lattice point w of T . Moreover, \mathcal{T}_w is a refinement of the trivial triangulation $\mathcal{T}_0 = \{T\}$. The main idea in proving the collinearity property of $\text{int}(T) \cap \mathbb{Z}^d$ is to start with the trivial triangulation $\mathcal{T}_0 = \{T\}$ and obtain a sequence $\mathcal{T}_1 \prec \dots \prec \mathcal{T}_k$ of refinements such that \mathcal{T}_k is a full triangulation of T and $|\mathcal{T}_k| \geq dk+1$, and then show that noncollinearity forces $|\mathcal{T}_k| > dk+1$. The following lemma appears as an assertion in the proof of (1) in [2]. We prove it here since it is crucial in computing $|\mathcal{T}_i| - |\mathcal{T}_{i-1}|$.

Lemma 3. *Suppose P is a clean, non-empty, lattice d -polytope. Let \mathcal{T} be a partial triangulation of P . If S is a j -face of some $T \in \mathcal{T}$, and $\mathcal{V}(S) \cap \text{int}(P) \cap \mathbb{Z}^d \neq \emptyset$, then S is contained in at least $d+1-j$ simplices in \mathcal{T} .*

Proof. Let $w \in \mathcal{V}(S) \cap \text{int}(P) \cap \mathbb{Z}^d$. By Lemma 1, S is contained in at least $d-j$ facets of T . Moreover w is a vertex of each facet of T containing S . It follows that these facets are not contained in facets of P and are therefore shared by exactly two simplices in \mathcal{T} . On the other hand any two simplices in \mathcal{T} intersect in at most one common facet. Thus S is contained in at least $d-j$ other simplices in \mathcal{T} . ■

Theorem 4. *Let P be a clean, non-empty, lattice d -polytope. Let \mathcal{T} be a partial triangulation of P . For any $w \in \text{int}(P) \cap \mathbb{Z}^d \setminus \mathcal{V}(\mathcal{T})$ there exists a refinement \mathcal{T}' of \mathcal{T} such that $\mathcal{V}(\mathcal{T}') = \mathcal{V}(\mathcal{T}) \cup \{w\}$ and $|\mathcal{T}'| \geq |\mathcal{T}| + d$.*

Proof. Since \mathcal{T} is a partial triangulation, there exists an interior lattice point w of P such that $w \notin \mathcal{V}(\mathcal{T})$. Moreover, w must lie in the relative interior of some j -face ($1 \leq j \leq d$), say S , of some simplex in \mathcal{T} . Note that S is in fact a j -simplex. Let $\mathcal{V}(S) = \{v_1, \dots, v_{j+1}\}$ and consider the basic triangulation \mathcal{T}_w of S into j -simplices, where

$$\mathcal{T}_w = \{ \text{conv}(\mathcal{V}(S) \cup \{w\} \setminus \{v_i\}) : 1 \leq i \leq j+1 \}.$$

Clearly \mathcal{T}_w is a refinement of S into j -simplices. This refinement of S induces a refinement of any d -simplex containing S . More precisely, if $T \in \mathcal{T}$ contains S , then the set

$$\{ \text{conv}(\mathcal{V}(T) \cup \{w\} \setminus \{v_i\}) \}$$

is a lattice triangulation of T . Since $w \in \text{int}(S)$ and P is clean, $S \not\subset \partial P$. Thus $\mathcal{V}(S) \cap \text{int}(P) \cap \mathbb{Z}^d \neq \emptyset$. By Lemma 3, there are at least $d+1-j$ simplices in \mathcal{T} containing S as a j -face. Now consider \mathcal{T} with all such d -simplices in \mathcal{T} replaced with their respective induced triangulations and take this to be \mathcal{T}' . By construction, w is contained in a simplex $T' \in \mathcal{T}'$ if and only if $w \in \mathcal{V}(T')$. Hence \mathcal{T}' is a refinement of \mathcal{T} such that $\mathcal{V}(\mathcal{T}') = \mathcal{V}(\mathcal{T}) \cup \{w\}$ and

$$|\mathcal{T}'| \geq |\mathcal{T}| + (d+1-j)(j+1) - (d+1-j) = |\mathcal{T}| + (d+1-j)j.$$

Finally,

$$(4) \quad (d+1-j)j - d = (d-j)(j-1) \geq 0 \quad \text{for } 1 \leq j \leq d,$$

which implies $(d+1-j)j \geq d$. ■

In the proof above, it is important to note that equality in (4) holds if and only if $j = d$ or $j = 1$. Equally important is that if $\mathcal{V}(\mathcal{T}') \neq P \cap \mathbb{Z}^d$, then \mathcal{T}' is again a partial triangulation.

Corollary 5. *If $T \in \mathcal{S}_k^d$ and $\mathcal{T}_0 = \{T\}$, then there exists a sequence*

$$(5) \quad \mathcal{T}_0 \prec \cdots \prec \mathcal{T}_k$$

of refinements of \mathcal{T}_0 such that \mathcal{T}_k is a full triangulation of T and $|\mathcal{T}_k| \geq dk + 1$. Moreover, $\text{Vol}(T) \geq \frac{1}{d!}(dk + 1)$.

Proof. For $k = 0$, then the sequence consists only of \mathcal{T}_0 . If $k > 0$ then $\mathcal{T}_0 = \{T\}$ is a partial triangulation of T . Let w_1, \dots, w_k be an arbitrary enumeration of the interior lattice points of T . For $1 \leq i \leq k$, we refine \mathcal{T}_{i-1} into \mathcal{T}_i , using Theorem 4 with $w = w_i$. After k refinements, \mathcal{T}_k is a full triangulation of T , and $|\mathcal{T}_k| \geq |\mathcal{T}_0| + dk = dk + 1$. Lastly, (2) implies $\text{Vol}(T) \geq \frac{1}{d!}(dk + 1)$. ■

Corollary 6. *Suppose $T \in \mathcal{S}_k^d$ where $k \geq 2$. Let w_1, \dots, w_k be an arbitrary enumeration of $\text{int}(T) \cap \mathbb{Z}^d$, and let (5) be the corresponding refinement sequence guaranteed by Corollary 5. If w_{i+1} lies in a j -face of a simplex in \mathcal{T}_i , where $i > 0$ and $1 < j < d$, then $|\mathcal{T}_{i+1}| - |\mathcal{T}_i| > d$ and $\text{Vol}(T) > \frac{1}{d!}(dk + 1)$.*

Proof. This follows immediately from Theorem 4 and (3), and (4). ■

In the context of Corollary 5, (4) implies that for each \mathcal{T}_i in (5), $|\mathcal{T}_i| \geq di + 1$. Since equality in (4) holds if and only if $j = 1$ or $j = d$, $|\mathcal{T}_i| = di + 1$ if and only if for each i , w_{i+1} lies in the relative interior of an edge in \mathcal{T}_i , or in the relative interior of a simplex in \mathcal{T}_i . Note that this must be true for any ordering of the w_i 's.

Finally, to prove (1) for a general non-empty lattice d -polytope P , we start with a basic triangulation of P and obtain a refinement sequence similar to that of Corollary 5.

Corollary 7. [2, Theorem 1.2] *If P is a lattice d -polytope with $k \geq 0$ interior lattice points, then there exists a sequence*

$$\mathcal{T}_1 \prec \cdots \prec \mathcal{T}_k$$

of lattice triangulations of P such that \mathcal{T}_1 is a basic triangulation of P , \mathcal{T}_k is a full triangulation of P and $|\mathcal{T}_k| \geq dk + 1$. Moreover, $\text{Vol}(P) \geq \frac{1}{d!}(dk + 1)$.

Proof. If $k = 0$, take \mathcal{T}_1 to be the triangulation guaranteed by Theorem 2. If $k \geq 1$, then consider an arbitrary enumeration w_1, \dots, w_k of the interior points of P . Let \mathcal{T}_1 be the basic triangulation \mathcal{T}_{w_1} of T . Either $k = 1$ and \mathcal{T}_1 is a full triangulation, or we can apply Theorem 4, as in the proof of Corollary 5, to obtain a refinement sequence

$$\mathcal{T}_1 \prec \cdots \prec \mathcal{T}_k$$

such that $|\mathcal{T}_k| \geq dk + 1$. Finally, (2) implies $\text{Vol}(P) \geq \frac{1}{d!}(dk + 1)$. ■

In general, we need not start with a basic triangulation of P . It is easy to check that so long as \mathcal{T} is a partial triangulation such that

$$|P \cap \mathbb{Z}^d| - |\mathcal{V}(\mathcal{T})| = k - j \quad \text{and} \quad |\mathcal{T}| \geq dj + 1,$$

then we can still refine \mathcal{T} into a full triangulation with at least $dk + 1$ simplices by applying Theorem 4. This is in fact equivalent to the inductive step in [2], with triangulations replaced by decompositions into d -subpolytopes. The proof of Corollary 7 is otherwise essentially the same as that in [2].

4. PROPERTIES OF $d + 2$ POINTS IN \mathbb{R}^d

In the context of Corollary 6, if we can show $j = d$ implies $\text{Vol}(T) > \frac{1}{d!}(dk + 1)$ as well, then any two interior lattice points of T must be collinear with some vertex of T . As a base case, we first consider the possible configurations of any $T \in \mathcal{S}_2^d$, where $d \geq 3$. Suppose T has interior lattice points w_1 and w_2 , and let $\mathcal{V}(T) = \{v_1, \dots, v_{d+1}\}$. The sets

$$(6) \quad \mathcal{P}_i = \mathcal{V}(T) \setminus \{v_i\} \cup \{w_1, w_2\}$$

are all sets of $d + 2$ points not contained in a hyperplane. Many properties of such sets are discussed in [3], [8], [13], [15], [17], [18], and [19]. Two relevant and well known results in the theory of convex bodies are Carathéodory's theorem [4] and Radon's theorem [19].

Carathéodory's Theorem. *If $\mathcal{P} = \{v_1, \dots, v_{d+1}\} \subset \mathbb{R}^d$ is not contained in a hyperplane, then every $x \in \mathbb{R}^d$ can be expressed as*

$$x = \sum_{i=1}^{d+1} \alpha_i v_i, \quad \text{where } \alpha_i \in \mathbb{R}, \quad v_i \in \mathcal{P}, \quad \text{and} \quad \sum_{i=1}^{d+1} \alpha_i = 1.$$

The coefficients α_i in Carathéodory's theorem are the *barycentric coordinates* of x relative to $\text{conv}(\mathcal{P})$. If $P = \text{conv}(\mathcal{P})$ is a d -simplex, then the barycentric coordinates of x relative to the simplex P are the numbers $\alpha_1, \dots, \alpha_{d+1}$ satisfying

$$\underbrace{\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{d+1} \end{bmatrix}}_{1 \times (d+1)} \cdot \underbrace{\begin{bmatrix} v_1 & 1 \\ v_2 & 1 \\ \vdots & \vdots \\ v_{d+1} & 1 \end{bmatrix}}_{(d+1) \times (d+1)} = \underbrace{\begin{bmatrix} x & 1 \end{bmatrix}}_{1 \times (d+1)}.$$

For each i , the sign of α_i indicates the position of x relative to the hyperplane H_i containing the facet of P opposite vertex v_i . That is, $\alpha_i > 0$ when v_i and x are on the same side of H_i , $\alpha_i < 0$ if v_i and x are on opposite sides of H_i , and $\alpha_i = 0$ if x lies in H_i . The barycentric coordinates of x relative to T are all positive if and only if $x \in \text{int}(P)$. Thus any point $x \in \mathcal{P}_i$ (cf. (6)), can be described in terms of its barycentric coordinates relative to the simplex $\text{conv}(\mathcal{P}_i)$.

Radon's Theorem. *If \mathcal{A} is a set of $k \geq d + 2$ points in \mathbb{R}^d , then there exist a partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ of \mathcal{A} such that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\text{conv}(\mathcal{A}_1) \cap \text{conv}(\mathcal{A}_2) \neq \emptyset$.*

The partition in Radon's theorem is called a *Radon partition* of \mathcal{A} . A Radon partition *in* \mathcal{A} is a Radon partition of a subset of \mathcal{A} . Let $\{\mathcal{A}_1, \mathcal{A}_2\}$ and $\{\mathcal{A}'_1, \mathcal{A}'_2\}$ be Radon partitions of \mathcal{A} . Then $\{\mathcal{A}_1, \mathcal{A}_2\}$ *extends* $\{\mathcal{A}'_1, \mathcal{A}'_2\}$ provided $\mathcal{A}'_i \subseteq \mathcal{A}_i$. In [8], Hare and Kennely introduced the notion of a *primitive Radon partition*, a Radon partition that is minimal with respect to extension. An immediate consequence is that if $\{\mathcal{A}_1, \mathcal{A}_2\}$ is a Radon partition of \mathcal{A} , then there exists a primitive Radon partition in \mathcal{A} such that $\{\mathcal{A}_1, \mathcal{A}_2\}$ extends it. Breen proved in [3] that $\{\mathcal{A}_1, \mathcal{A}_2\}$ is a primitive Radon partition in \mathcal{A} if and only if $\mathcal{A}_1 \cup \mathcal{A}_2$ is in general position in $\mathbb{R}^{|\mathcal{A}_1|+|\mathcal{A}_2|}$. Recall that $\mathcal{A}_1 \cup \mathcal{A}_2$ is in general position if no $j + 2$ points in this union lie in a j -flat for all $j < |\mathcal{A}_1| + |\mathcal{A}_2|$. Peterson proved in [17] that the Radon partition of $d + 2$ points in general position in \mathbb{R}^d is unique, and both Breen and Peterson showed that if $\mathcal{A}_1 \cup \mathcal{A}_2$ is in general position in $\mathbb{R}^{|\mathcal{A}_1|+|\mathcal{A}_2|}$, then $\text{conv}(\mathcal{A}_1) \cap \text{conv}(\mathcal{A}_2)$

is a single point. Lastly, Proskuryakov proved in [18] that if $\mathcal{P} \subset \mathbb{R}^d$ is a set of $d+2$ points in general position, then two points will lie in the same component of the (unique) Radon partition of P if and only if they are separated by the hyperplane through the remaining d points. Kosmak also proved this result in [13] using affine varieties. These properties of Radon partitions are equivalent to Lawson's First and Second Theorems from [15].

Lawson's First Theorem. [15, Theorem 1] *Let $\mathcal{P} = \{v_1, \dots, v_{d+2}\} \subset \mathbb{R}^d$ and suppose \mathcal{P} does not lie entirely in any hyperplane. There is a partition of \mathcal{P} into three sets \mathcal{A}_0 , \mathcal{A}_1 , and \mathcal{A}_2 , and $\alpha_i \in \mathbb{R}$, satisfying*

$$\begin{aligned} (7) \quad \sum_{v_i \in \mathcal{A}_1} \alpha_i v_i &= \sum_{v_i \in \mathcal{A}_2} \alpha_i v_i, \\ (8) \quad \sum_{v_i \in \mathcal{A}_1} \alpha_i &= \sum_{v_i \in \mathcal{A}_2} \alpha_i = 1, \\ (9) \quad \alpha_i &> 0 \quad \text{if } v_i \in \mathcal{A}_1 \cup \mathcal{A}_2. \end{aligned}$$

The numbers α_i are uniquely determined by the set \mathcal{P} . We set $\alpha_i = 0$ if $v_i \in \mathcal{A}_0$. The sets \mathcal{A}_0 and $\{\mathcal{A}_1, \mathcal{A}_2\}$ are also unique.

Since the sets \mathcal{A}_1 and \mathcal{A}_2 in Lawson's First Theorem form the unique primitive Radon partition in \mathcal{P} , the point

$$\text{conv}(\mathcal{A}_1) \cap \text{conv}(\mathcal{A}_2) = \sum_{v_i \in \mathcal{A}_1} \alpha_i v_i = \sum_{v_i \in \mathcal{A}_2} \alpha_i v_i,$$

is in the relative interior of $\text{conv}(\mathcal{A}_1)$ and $\text{conv}(\mathcal{A}_2)$ by (7), (8), and (9). These sets also determine the possible triangulations of $\text{conv}(\mathcal{P})$.

Lawson's Second Theorem. [15, Theorem 2] *Let $\mathcal{P} = \{v_1, \dots, v_{d+2}\} \subset \mathbb{R}^d$, and let T_i be the simplex with vertex set $\mathcal{V}(T_i) = \mathcal{P} \setminus \{v_i\}$. There are at most two distinct triangulations of $P = \text{conv}(\mathcal{P})$, namely*

$$\mathcal{T}_1 = \{T_i : v_i \in \mathcal{A}_1\} \quad \text{and} \quad \mathcal{T}_2 = \{T_i : v_i \in \mathcal{A}_2\},$$

where the sets \mathcal{A}_1 and \mathcal{A}_2 are as defined in Lawson's First Theorem. The set \mathcal{T}_j is a valid triangulation if and only if $|\mathcal{A}_j| > 1$, where $j \in \{1, 2\}$.

Corollary 8. [15, Corollary 1] *In the context of building triangulations, an enumeration of all possible configurations of $d+2$ points in \mathbb{R}^d , not lying in any hyperplane, is given by all of the possible ways of assigning values to $|\mathcal{A}_0|$, $|\mathcal{A}_1|$, and $|\mathcal{A}_2|$ satisfying*

$$\begin{aligned} (10) \quad |\mathcal{A}_2| &\geq |\mathcal{A}_1| \geq 1, \\ (11) \quad |\mathcal{A}_2| &\geq 2, \\ (12) \quad |\mathcal{A}_0| &\geq 0, \\ (13) \quad |\mathcal{A}_0| + |\mathcal{A}_1| + |\mathcal{A}_2| &= d+1. \end{aligned}$$

Consider the possible sets of five points in \mathbb{R}^3 such that the convex hull is a non-degenerate polyhedron with two different triangulations. Lawson's Second Theorem implies $|\mathcal{A}_1| \geq 2$. Inequalities (10), (11), (12), and (13) imply either

$$(|\mathcal{A}_0|, |\mathcal{A}_1|, |\mathcal{A}_2|) = (0, 2, 3) \quad \text{or} \quad (|\mathcal{A}_0|, |\mathcal{A}_1|, |\mathcal{A}_2|) = (1, 2, 2).$$

Example 2. Suppose $\mathcal{P} = \{v_1, \dots, v_5\} \subset \mathbb{R}^3$ is in general position. As shown on the left in Figures 2 and 3, let $P = \text{conv}(\mathcal{P})$. In the context of Lawson's First Theorem, $\mathcal{A}_0 = \emptyset$, $\mathcal{A}_1 = \{v_1, v_5\}$, and $\mathcal{A}_2 = \{v_2, v_3, v_4\}$. The polyhedron P is a “bipyramid” with triangulations

$$\mathcal{T}_1 = \{ \text{conv}(v_1, v_2, v_3, v_4), \text{conv}(v_2, v_3, v_4, v_5) \}$$

and

$$\mathcal{T}_2 = \{ \text{conv}(v_1, v_2, v_3, v_5), \text{conv}(v_1, v_2, v_4, v_5), \text{conv}(v_1, v_3, v_4, v_5) \}$$

by Lawson's Second Theorem.

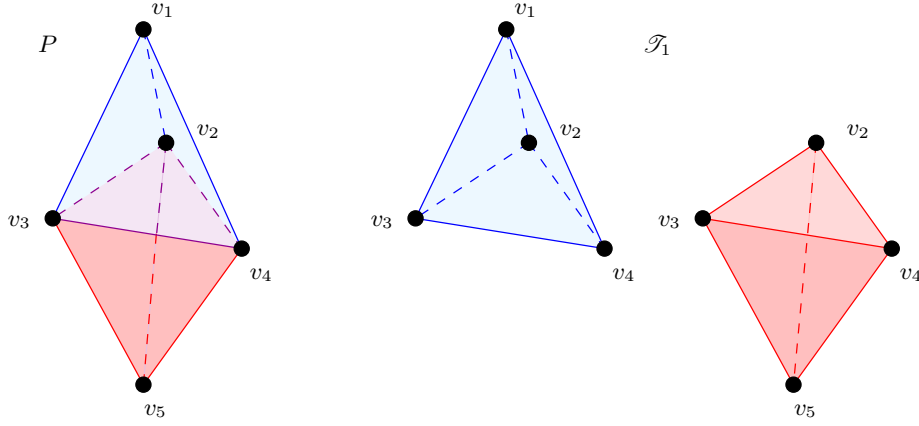


FIGURE 2. P (left) and the two simplices of \mathcal{T}_1 (right)

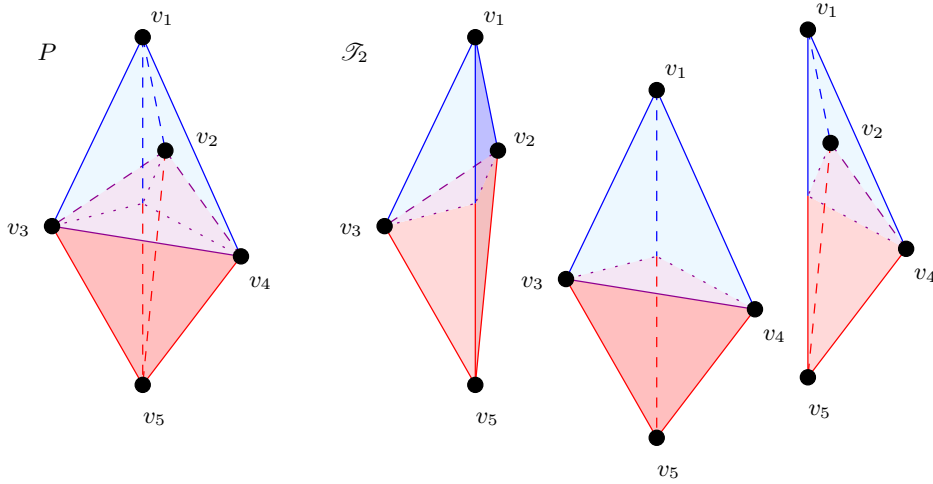


FIGURE 3. P (left) and the three simplices of \mathcal{T}_2 (right)

Example 3. Consider $\mathcal{P} = \{v_1, \dots, v_5\} \subset \mathbb{R}^3$ and $P = \text{conv}(\mathcal{P})$ shown on the left in Figure 4. The polyhedron P is a nondegenerate pyramid whose base is a planar quadrilateral with vertex set $\{v_2, v_3, v_4, v_5\}$. In the context of Lawson's First Theorem, $\mathcal{A}_0 = \{v_1\}$, $\mathcal{A}_1 = \{v_2, v_4\}$ and $\mathcal{A}_2 = \{v_3, v_5\}$. Lawson's Second Theorem implies P has two triangulations

$$\mathcal{T}_1 = \{ \text{conv}(v_1, v_3, v_4, v_5), \text{conv}(v_1, v_2, v_3, v_5) \}$$

and

$$\mathcal{T}_2 = \{ \text{conv}(v_1, v_2, v_4, v_5), \text{conv}(v_1, v_2, v_3, v_4) \}.$$

Each triangulation is induced by the triangulation (in dimension 2) of the base.

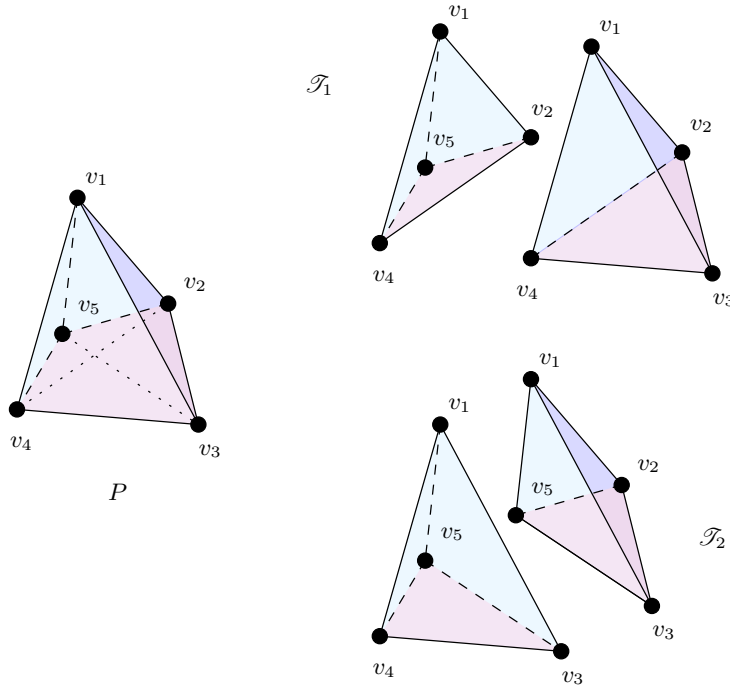


FIGURE 4. P (left) and both of its triangulations (right)

In both Examples 2 and 3, $|\mathcal{A}_1| = 2$, and $\text{conv}(\mathcal{A}_1 \cup \mathcal{A}_2)$ can be viewed as two simplices joined at a common facet. Moreover, the line segment formed by the vertices not in the common facet intersects that facet at a point. Such a configuration is called a bipyramid. In general, a d -polytope P is a d -bipyramid if P is the convex hull of a line segment L and a $(d-1)$ -simplex S such that the intersection $L \cap S$ is a single point contained in $\text{int}(L) \cap \text{int}(S)$. A 1-bipyramid is simply a line segment with a point in its interior (or the convex hull of three collinear points). In Example 2, P itself is a 3-bipyramid, whereas in Example 3 P contains a 2-bipyramid (its base). The following result is a direct consequence of Lawson's theorems.

Corollary 9. *Let P be a d -polytope with $d+2$ vertices. If there exists a subset $\mathcal{P} \subseteq \mathcal{V}(P)$ such that $\text{conv}(\mathcal{P})$ is a j -bipyramid, where $2 \leq j \leq d$, then $\mathcal{V}(P)$ has exactly triangulations, one with cardinality 2 and another with cardinality j .*

5. EXISTENCE OF BIPYRAMIDS IN $T \in \mathcal{S}_k^d$

For $k \geq 2$, under what conditions will $T \in \mathcal{S}_k^d$ contain a bipyramid? Consider the following construction of possible bipyramids from a d -simplex. Let S be a d -simplex, and let $\mathcal{V}(S) = \{v_1, \dots, v_{d+1}\}$. For any point x exterior to S , let (α_i) be the barycentric coordinates of x relative to S . By the definition of a bipyramid, $P = \text{conv}(S, x)$ is a bipyramid if and only if exactly one α_i is negative, and the remainder are positive. If we allow for, say, j of the α_i to be zero, then P will contain a $(d - j)$ -bipyramid.

The geometric interpretation is as follows. Let H_i be the hyperplane containing the facet of S opposite v_i . Then P is a d -bipyramid if x and S are on opposite sides of exactly one H_i , and x is on the same side of the remaining hyperplanes as S . If x is contained in j of these hyperplanes, then P contains a $(d - j)$ -bipyramid.

Theorem 10. Suppose $T \in \mathcal{S}_k^d$, where $k \geq 2$. Let w_1 and w_2 be any two interior lattice points of T , and let $\mathcal{T}_{w_1} = \{T_i\}$ be the basic triangulation of T with respect to w_1 . If w_2 lies in the relative interior of a simplex in \mathcal{T}_{w_1} , say T_n , then there exist numbers α_i such that

$$w_2 = \alpha_1 w_1 + \alpha_n v_n + \sum_{\substack{i=2 \\ i \neq n}}^{d+1} \alpha_i v_i, \quad \sum_{i=1}^{d+1} \alpha_i = 1,$$

$\alpha_1 > 0$, $\alpha_n < 0$, and $\alpha_i \geq 0$ otherwise.

Proof. Let $\mathcal{V}(T) = \{v_1, \dots, v_{d+1}\}$ and F_i be the facet of T opposite the vertex v_i . We may assume without loss of generality that $T_i = \text{conv}(F_i, w_1)$ and $n = d + 1$ (i.e. $w_2 \in \text{int}(T_{d+1})$). Let L be the line through v_{d+1} and w_1 .

Case 1: If w_2 lies on L , then w_1 is necessarily between v_{d+1} and w_2 . There exists $\alpha > 1$ such that

$$w_2 = v_{d+1} + \alpha(w_1 - v_{d+1}) = \alpha w_1 + (1 - \alpha)v_{d+1}.$$

Choose $\alpha_1 = \alpha$, $\alpha_{d+1} = 1 - \alpha$, and $\alpha_i = 0$ for $2 \leq i \leq d + 1$.

Case 2: Suppose w_2 does not lie on L . Let $x = L \cap F_{d+1}$. If $x \notin \text{int}(F_{d+1})$, then the line segment $v_{d+1}x$, which contains w_1 , would be contained in a facet of T and contradict the cleanliness of T . Thus $x \in \text{int}(F_{d+1})$ and consequently x cannot be a lattice point. Since $x \in \text{int}(F_{d+1})$ if and only if there exists β_i such that

$$x = \sum_{i=1}^d \beta_i v_i, \quad \beta_i > 0, \quad \text{and} \quad \sum_{i=1}^d \beta_i = 1,$$

the set $\mathcal{P} = \mathcal{V}(T_{d+1}) \cup \{x\} \not\subset \mathbb{Z}^d$ can be partitioned into

$$\mathcal{A}_0 = \{w_1\}, \quad \mathcal{A}_1 = \{x\}, \quad \text{and} \quad \mathcal{A}_2 = \mathcal{V}(F_{d+1})$$

according to Lawson's First Theorem. By Lawson's Second Theorem, \mathcal{P} has exactly one (non-lattice) triangulation

$$\mathcal{T} = \{ \text{conv}(\mathcal{V}(T_{d+1}) \setminus \{v_i\} \cup \{x\}) : 1 \leq i \leq d \}.$$

Since $T_{d+1} = \text{conv}(\mathcal{P})$ and $w_2 \in \text{int}(T_{d+1})$, w_2 must be in some (non-lattice) simplex in \mathcal{T} . Without loss of generality, suppose

$$w_2 \in \text{conv}(v_2, \dots, v_d, w_1, x).$$

Then there exist γ_i such that $\gamma_i \geq 0$ for $1 \leq i \leq d+1$,

$$\sum_{i=1}^{d+1} \gamma_i = 1, \quad \text{and} \quad w_2 = \gamma_1 w_1 + \sum_{i=2}^d \gamma_i v_i + \gamma_{d+1} x.$$

Since w_2 cannot lie in any face of T , $\gamma_1 > 0$. Similarly, $\gamma_{d+1} > 0$ since w_2 is not in any face of T_{d+1} . Furthermore, the assumption that w_2 does not lie on L implies one of the remaining γ_i ($2 \leq i \leq d$) must also be positive. Since w_1 lies between v_{d+1} and x on L , there exists $\mu > 1$ such that

$$x = v_{d+1} + \mu(w_1 - v_{d+1}) = \mu w_1 + (1 - \mu)v_{d+1}.$$

It follows that

$$w_2 = (\gamma_1 + \mu\gamma_{d+1})w_1 + \sum_{i=2}^d \gamma_i v_i + (1 - \mu)\gamma_{d+1}v_{d+1}.$$

Let

$$\alpha_1 = \gamma_1 + \mu\gamma_{d+1}, \quad \alpha_{d+1} = (1 - \mu)\gamma_{d+1}, \quad \text{and} \quad \alpha_i = \gamma_i \quad \text{for} \quad 2 \leq i \leq d.$$

Note that

$$\sum_{i=1}^{d+1} \alpha_i = [\mu + (1 - \mu)]\gamma_{d+1} + \sum_{i=1}^d \gamma_i = 1,$$

$\alpha_1 > 0$, and $\alpha_{d+1} < 0$. The remaining α_i are nonnegative, and at least one is positive since at least one of the γ_i is positive for $2 \leq i \leq d$. \blacksquare

The geometric interpretation of Theorem 10 is that for some simplex $T_m \neq T_n$ in \mathcal{T}_{w_1} , $\text{conv}(T_m, w_2)$ contains a j -bipyramid P , and $\mathcal{V}(P)$ includes w_1 and w_2 . In terms of triangulations, $\mathcal{V}(T) \cup \{w_1, w_2\}$ has two triangulations, provided $j > 1$.

Corollary 11. *Suppose $T \in \mathcal{S}_k^d$, where $k \geq 2$. If w_2 lies in the relative interior of a simplex in the basic triangulation \mathcal{T}_{w_1} of T , then the set $\mathcal{V}(T) \cup \{w_1, w_2\}$ has two triangulations provided w_1 and w_2 are not collinear with any $v \in \mathcal{V}(T)$.*

Proof. Let $\mathcal{T}_1 = \mathcal{T}_{w_1} = \{T_i\}$. Without loss of generality, suppose $w_2 \in \text{int}(T_{d+1})$. Let \mathcal{T}_{w_2} be the basic triangulation of T_{d+1} with respect to w_2 . One triangulation of $\mathcal{V}(T) \cup \{w_1, w_2\}$ is the refinement \mathcal{T}_2 of \mathcal{T}_1 guaranteed by Theorem 4 (with $w = w_2$), where

$$\mathcal{T}_2 = \mathcal{T}_{w_1} \setminus \{T_{d+1}\} \cup \mathcal{T}_{w_2}.$$

Let S_i be the facet of T_{d+1} opposite v_i for $1 \leq i \leq d$, and let S_{d+1} be the facet of T_{d+1} opposite w_1 . Theorem 10 guarantees there exists $1 \leq m \leq d$ such that

$$\text{conv}(T_m, w_2) = \text{conv}\left(T_m \bigcup \text{conv}(S_m, w_2)\right).$$

contains a j -bipyramid. Note that both T_m and $\text{conv}(S_m, w_2)$ are simplices in \mathcal{T}_2 . Since w_1 and w_2 are not collinear with any $v \in \mathcal{V}(T)$, T is clean, and no three vertices of T are collinear, it follows that $j > 1$. Corollary 9 implies $\mathcal{V}(T_m) \cup \{w_2\}$ has two triangulations, one of which is contained in \mathcal{T}_2 . That is,

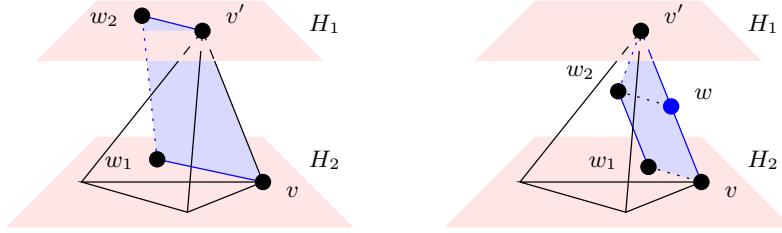
$$\text{conv}(T_m, w_2) = T_m \bigcup \text{conv}(S_m, w_2).$$

Thus $\mathcal{V}(T) \cup \{w_1, w_2\}$ has two triangulations. \blacksquare

Among all possible d -bipyramids that have two triangulations, 2-bipyramids are unique in that their triangulations have the same cardinality. The existence of lattice 2-bipyramids within lattice d -simplices has further implications. Note that 2-bipyramids are simply convex planar quadrilaterals.

Lemma 12. *Suppose $T \in S_k^d$ where $k \geq 2$. Let w_1 and w_2 be any two of the interior points of T . Let $Q = \text{conv}(w_1, w_2, v, v')$ where $v, v' \in \mathcal{V}(T)$. If Q is a planar quadrilateral, then the opposing edges of Q cannot be parallel.*

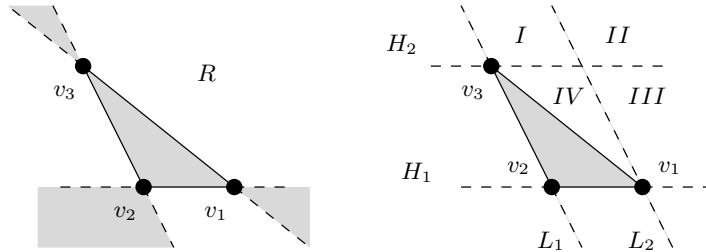
Proof. Without loss of generality, suppose w_1v and w_2v' are edges of Q . Let H_1 be the hyperplane containing the facet of T opposite v' , and let H_2 be the hyperplane parallel to H_1 and containing v' . Since w_1 is an interior point of T , it must lie between H_1 and H_2 . If $w_1v \parallel w_2v'$, then w_2 must be opposite of w_1 relative to both



H_1 and H_2 . This is impossible since w_2 would lie outside of T as shown in the left figure above. If $w_1w_2 \parallel vv'$, edge w_1w_2 of Q must be shorter than edge vv' since w_1w_2 is in the interior of T . Then $w = v' - (w_2 - w_1)$, or $w = v + (w_2 - w_1)$, is a lattice point on the edge vv' , as shown above on the right, which is also impossible as T is clean. ■

Lemma 13. *Suppose $Q = \text{conv}(v_1, v_2, v_3, v_4)$ is a planar lattice quadrilateral. If the opposing edges of Q are not parallel, then interior of Q contains a lattice point $w \neq v_i$ for $1 \leq i \leq 4$. Moreover, w lies in the interior of a triangle whose edge set is a subset of the edges and diagonals of Q .*

Proof. Consider the pairs of adjacent edges of Q . Without loss of generality, suppose v_1v_2 and v_2v_3 are edges of Q such that the triangle $\Delta = \text{conv}(v_1, v_2, v_3)$ has minimal area. The lines containing each edge of Δ partition the plane into 7 different regions as shown below (left). Since Q is a convex quadrilateral, v_4 can



only lie in the three unshaded regions. Without loss of generality, suppose v_4 lies in region R as shown above. Let H_1 be the line through v_1v_2 and H_2 be the line through v_3 such that $H_2 \parallel H_1$. Similarly, let L_1 be the line through v_2v_3 and L_2

be the line through v_1 such that $L2 \parallel L1$. (See right-side figure above.) These four lines divide R into four regions. Since Q has no parallel edges, v_4 cannot lie on any one of these four lines. If v_4 lies in region I, then $\text{conv}(v_2, v_3, v_4)$ is a triangle of smaller area than Δ (they both share the same leg v_2v_3 , whereas the distance between v_4 and L_1 is shorter than the distance between v_1 and L_1). Similarly, v_4 cannot lie in regions III or IV. Thus v_4 must lie in region II. Then $w = v_1 + (v_3 - v_2)$ is a lattice point contained in interior of $\text{conv}(v_1, v_3, v_4)$. ■

6. COLLINEARITY PROPERTY OF MINIMAL VOLUME $T \in \mathcal{S}_k^d$

We can now prove the collinearity property of $\text{int}(T) \cap \mathbb{Z}^d$, where $T \in \mathcal{S}_k^d$, $d \geq 3$, and $\text{Vol}(T) = \frac{1}{d!}(dk + 1)$.

Theorem 14. *Suppose $T \in \mathcal{S}_k^d$, where $d \geq 3$ and $\text{Vol}(T) = \frac{1}{d!}(dk + 1)$. Any two interior lattice points of T are collinear with a vertex of T .*

Proof. The claim is vacuously true for $k \leq 1$, so suppose $k \geq 2$. If $k = 2$ then let w_1 and w_2 be the two interior lattice points of T . Otherwise let w_1 and w_2 be any two interior lattice points of T . Suppose w_1 and w_2 are not collinear with any $v \in \mathcal{V}(T)$. That is, suppose there does not exist a lattice 1-bipyramid consisting of w_1 , w_2 , and any $v \in \mathcal{V}(T)$. Let $\mathcal{T}_1 = \mathcal{T}_{w_1} = \{T_i\}$, where \mathcal{T}_{w_1} is the basic triangulation of T with respect to w_1 . By assumption, w_2 must lie in the relative interior of either a simplex in \mathcal{T}_1 , or a j -face of a simplex in \mathcal{T}_1 , where $1 < j < d$.

Case 1: If w_2 lies in the relative interior of a j -face of a simplex in \mathcal{T}_1 , where $1 < j < d$, then Corollary 6 implies $\text{Vol}(T) > \frac{1}{d!}(dk + 1)$, a contradiction.

Case 2: If w_2 lies in the relative interior of a simplex in \mathcal{T}_1 , then we may assume without loss of generality that $w \in \text{int}(T_{d+1})$. Let \mathcal{T}_{w_2} be the basic triangulation of T_{d+1} with respect to w_2 . The refinement \mathcal{T}_2 of \mathcal{T}_1 obtained by applying Theorem 4 with $w = w_2$ is

$$\mathcal{T}_2 = \mathcal{T}_{w_1} \setminus \{T_{d+1}\} \cup \mathcal{T}_{w_2}.$$

Theorem 10 implies T contains a lattice j -bipyramid P , where $2 \leq j \leq d$ and $\{w_1, w_2\} \subset \mathcal{V}(P)$. If $j = 2$, then $P = \text{conv}(v, v', w_1, w_2)$ is convex planar lattice quadrilateral, where $\{v, v'\} \subset \mathcal{V}(T)$. Lemmas 12 and 13 imply P contains a lattice point $w_3 \notin \{w_1, w_2\}$. This is a contradiction if $k = 2$. If $k > 2$, then the second part of Lemma 13 implies w_3 lies in the relative interior of some triangle Δ_i , where

$$\begin{aligned} \Delta_1 &= \text{conv}(w_1, w_2, v), & \Delta_2 &= \text{conv}(w_1, w_2, v'), \\ \Delta_3 &= \text{conv}(w_1, v, v'), & \Delta_4 &= \text{conv}(w_2, v, v'). \end{aligned}$$

Since $d \geq 3$, and each Δ_i a 2-face of simplices in \mathcal{T}_2 , $\text{Vol}(T) > \frac{1}{d!}(dk + 1)$ by Corollary 6. This contradicts the minimal volume property of T . On the other hand, if $j > 2$, then $\mathcal{V}(T) \cup \{w_1, w_2\}$ has two possible triangulations \mathcal{T}_2 and \mathcal{T}_2' where

$$|\mathcal{T}_2'| = |\mathcal{T}_1| + d - 2 + j > 2d + 1.$$

Subsequent applications of Theorem 4, starting with \mathcal{T}_2' , and (2) imply

$$\text{Vol}(T) > \frac{1}{d!}(dk + 1).$$

This is again a contradiction. Thus any two points in $\text{int}(T) \cap \mathbb{Z}^d$ must be collinear with some $v \in \mathcal{V}(T)$. ■

Corollary 15. *Suppose $T \in \mathcal{S}_k^d$, where $d \geq 3$ and $\text{Vol}(T) = \frac{1}{d!}(dk+1)$. The points $\text{int}(T) \cap \mathbb{Z}^d$ are collinear with some $v \in \mathcal{V}(T)$.*

Proof. The claim is vacuously true for $k \leq 1$. Theorem 14 implies the claim also holds for $k = 2$. Suppose $k \geq 3$. It suffices to show the claim holds for any three points in $\text{int}(T) \cap \mathbb{Z}^d$. Let w_1, w_2 , and w_3 be any three points in $\text{int}(T) \cap \mathbb{Z}^d$. Theorem 14 implies there exists $v \in \mathcal{V}(T)$ such that w_1, w_2 , and v are collinear. Similarly, there exists $v' \in \mathcal{V}(T)$ such that w_1, w_3 , and v' are collinear. We need to show $v = v'$. Suppose $v \neq v'$. Let L be the line through w_1, w_2 and v , and let L' be the line through w_1, w_3 , and v' . Without loss of generality, suppose w_2 is between w_1 and v on L .

Case 1: If w_3 lies between w_1 and v' on L' , then $P = \text{conv}(v, v', w_2, w_3)$ is a convex planar quadrilateral contained within $\text{conv}(w_1, v, v')$. Lemmas 12 and 13 imply $\text{int}(P)$ contains a lattice point $w_4 \notin \{w_1, w_2, w_3\}$. Since $P \subset \text{conv}(w_1, v, v')$ and $d \geq 3$, Corollary 6 implies $\text{Vol}(T) > \frac{1}{d!}(dk+1)$.

Case 2: If w_1 lies between w_3 and v' on L' , then $w_2 \in \text{int}(\text{conv}(w_3, v, v'))$. Again, since $d \geq 3$, Corollary 6 implies $\text{Vol}(T) > \frac{1}{d!}(dk+1)$. Thus $v = v'$, and all points in $\text{int}(T)$ and v are collinear. ■

Corollary 16. *Suppose $T \in \mathcal{S}_k^d$ and $d \geq 3$. Let L be the line through $\text{int}(T) \cap \mathbb{Z}^d$. The consecutive points on L are evenly spaced.*

Proof. Let $v = L \cap \mathcal{V}(T)$ and let $w \in \text{int}(T)$ be the lattice point closest to v . Since any line can be described as a linear combination of two points, any $w' \in L$ can be expressed as

$$(14) \quad w' = v + \alpha(w - v),$$

where $\alpha \in \mathbb{R}$. It suffices to show that for any $w' \in L \cap T \cap \mathbb{Z}^d$, the α in (14) is an integer and $0 \leq \alpha \leq k$. If $\alpha < 0$, then $v \in \text{int}(\text{conv}(w, w')) \subset \text{int}(T)$, which is impossible. If $\alpha = 0$ then $w' = v$. Let $[a]$ and $\{\alpha\}$ denote the integer and fractional parts of α , respectively. If $\{\alpha\} \neq 0$, then

$$x = w' - [\alpha](w - v) = v + \{\alpha\}(w - v) \in \mathbb{Z}^d$$

lies between v and w on L , which contradicts our assumption that w is closest to v on L . Hence $\alpha \in \mathbb{Z}$. Finally, if $\alpha > k$, then T has more than k interior points, which is also impossible. ■

Corollary 17. *If $T \in \mathcal{S}_k^d$, $\text{Vol}(T) = \frac{1}{d!}(dk+1)$, and $d \geq 3$, then $T \cap \mathbb{Z}^d$ has a unique triangulation.*

Proof. Let $\mathcal{V}(T) = \{v_1, \dots, v_{d+1}\}$. Without loss of generality, suppose v_{d+1} is collinear with $\text{int}(T) \cap \mathbb{Z}^d$. Let w_1 be the point in $\text{int}(T) \cap \mathbb{Z}^d$ closest to v_{d+1} and

$$\mathcal{W} = \{w_i : w_i = v_{d+1} + i(w_1 - v_{d+1}), 0 \leq i \leq k\}.$$

Let $S_{k+1,d} = \text{conv}(v_1, \dots, v_d, w_k)$. For $1 \leq i \leq k$ and $1 \leq j \leq d$, let

$$S_{i,j} = \text{conv}(\mathcal{V}(T) \setminus \{v_j, v_{d+1}\} \cup \{w_i, w_{i-1}\}).$$

It is easy to check that $\mathcal{T} = \{S_{i,j}\}$ is a triangulation of $T \cap \mathbb{Z}^d$. In fact, this triangulation corresponds to the refinement sequence in Corollary 5.

Let \mathcal{T}' be another full triangulation of T . Consider any $S \in \mathcal{T}'$. Since \mathcal{T}' is a full triangulation, S is necessarily empty and clean. These two conditions force

$$1 \leq |\mathcal{V}(S) \cap \mathcal{W}| \leq 2.$$

Case 1: If $|\mathcal{V}(S) \cap \mathcal{W}| = 1$, then there exist $v \in \mathcal{V}(T)$ and $w \in \mathcal{W}$ such that

$$S = \text{conv}(\mathcal{V}(T) \setminus \{v\} \cup \{w\}).$$

Since S is empty, it follows that $w = w_k$, $v = v_{d+1}$, and $S = S_{k+1,d}$.

Case 2: If $|\mathcal{V}(S) \cap \mathcal{W}| = 2$, then there exist $\{v, v'\} \subset \mathcal{V}(T)$ and $\{w, w'\} \subset \mathcal{W}$ such that

$$S = \text{conv}(\mathcal{V}(T) \setminus \{v, v'\} \cup \{w, w'\}).$$

Without loss of generality, suppose w is closer to w_0 than w' is to w_0 . Since S is clean,

$$(15) \quad (w, w') = (w_{i-1}, w_i)$$

for some $1 \leq i \leq k$. Moreover, $w_0 (= v_{d+1})$ is a vertex of S if and only if $i = 1$ in (15) since S must also be empty. Thus $S = S_{i,j}$ for some appropriate j , and $\mathcal{T}' \subseteq \mathcal{T}$. Consequently, $\mathcal{T}' = \mathcal{T}$. \blacksquare

7. UNIMODULAR TRANSFORMATIONS

A *unimodular transformation* $f : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ is an affine map of the form

$$f(v) = v \cdot \mathbf{M} + u,$$

where $u, v \in \mathbb{Z}^d$, \mathbf{M} is a $d \times d$ matrix with entries in \mathbb{Z} , and $\det(\mathbf{M}) = \pm 1$. A *translation* is a unimodular transformation in which $\mathbf{M} = \mathbf{I}_d$, where \mathbf{I}_d is the $d \times d$ identity matrix. If $\det(\mathbf{M}) = \pm 1$, \mathbf{M} is invertible, then \mathbf{M}^{-1} is unimodular, and f is one-to-one. Thus P is a lattice d -polytope if and only if $f(P)$ is a lattice d -polytope with the same number of vertices. Using barycentric coordinates, it is easy to check that $f(w)$ is an interior point of $f(P)$ if and only if w is an interior point of P . Similarly, $f(w)$ is a boundary point of $f(P)$ if and only if w is a boundary point of P (cf. [23]). We say P_1 and P_2 are *equivalent* lattice d -polytopes (and write $P_1 \simeq P_2$) if there exists a unimodular transformation f such that $P_1 = f(P_2)$.

Lemma 18. *Let T be a d -simplex with $\text{Vol}(T) = \frac{1}{d!}$. Then T is equivalent to $T_{1,\dots,1}$, the d -simplex whose vertex set consists of the origin, e_i for $1 \leq i \leq d-1$, and the point $(1, \dots, 1)$.*

Proof. Let v_1, \dots, v_{d+1} be an enumeration of the vertices of T . If necessary, translate T so that one of its vertices is the origin. Without loss of generality, suppose $v_{d+1} = \mathbf{0}$. Let

$$P = \text{conv} \left(v_1, \dots, v_{d+1}, \sum_{i=1}^d v_i \right).$$

Note that P is a parallelepiped containing T , and $\text{Vol}(P) = \pm \det(\mathbf{A})$, where \mathbf{A} is a $d \times d$ matrix whose entry $a_{i,j}$ is the j -th coordinate of v_i . Since

$$\frac{1}{d!} = \text{Vol}(T) = \frac{1}{d!} \cdot \text{Vol}(P) = \frac{1}{d!} \cdot |\det(\mathbf{A})|,$$

$\det(\mathbf{A}) = \pm 1$ and \mathbf{A} is invertible. Thus, there exists a unique, unimodular, $d \times d$ matrix \mathbf{M} such that $\mathbf{AM} = \mathbf{I}_d$. Thus $f : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ defined by $f(v) = v \cdot \mathbf{M}$ is a unimodular transformation such that $f(v_i) = e_i$ for $1 \leq i \leq d$ and $f(v_{d+1}) = \mathbf{0}$. Let $g : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ be the unimodular transformation given by $g(v) = v \cdot \mathbf{M}'$ where

$$\mathbf{M}' = \begin{bmatrix} \mathbf{I}_{d-1} & 0 \\ 1 & 1 \end{bmatrix}.$$

It is easy to check that $g(e_i) = e_i$ for $1 \leq i \leq d-1$, $g(e_d) = (1, \dots, 1)$, and $g(\mathbf{0}) = \mathbf{0}$. Then the composition $g \circ f$, where $(g \circ f)(v) = v \cdot \mathbf{M} \cdot \mathbf{M}'$, is also a unimodular transformation, and $(g \circ f)(T) = T_{1, \dots, 1}$. \blacksquare

Let T_{a_1, \dots, a_d} denote the d -simplex whose vertex set consists of the origin, the unit points e_1, \dots, e_{d-1} , and the point $(a_1, \dots, a_d) \in \mathbb{Z}^d$. What can we say about T_{a_1, \dots, a_d} ? If $a_d < 0$, then we can apply $f : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$, where

$$(16) \quad f(v) = v \cdot \begin{bmatrix} \mathbf{I}_{d-1} & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that f is simply a reflection in the d -th coordinate. By reflection, if necessary, we can take a_d to be nonnegative. Moreover, a_d is determined by the volume of T_{a_1, \dots, a_d} since

$$(17) \quad 0 < d! \cdot \text{Vol}(T_{a_1, \dots, a_d}) = \pm \det \begin{bmatrix} a_1 & \dots & a_d \\ \mathbf{I}_{d-1} & & 0 \end{bmatrix} = |a_d|.$$

For the remaining a_i , consider the integers b_i such that

$$0 \leq a_i + b_i a_d < a_d,$$

and the transformation $g : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$, where

$$(18) \quad g(v) = v \cdot \left[\begin{array}{ccc|c} & & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline b_1 & \dots & b_{d-1} & 1 \end{array} \right].$$

All the vertices of T_{a_1, \dots, a_d} excluding (a_1, \dots, a_d) are fixed under g . On the other hand, g sends (a_1, \dots, a_d) to (a'_1, \dots, a'_d) where $a'_d = a_d$ and $a'_i = a_i + b_i a_d$ for $1 \leq i \leq d-1$. Since T_{a_1, \dots, a_d} is not contained in any hyperplane, $a_i > 0$ and $a'_i > 0$. Thus the class of all $T_{a_1, \dots, a_d} \in \mathcal{S}_k^d$ is represented by $T_{a'_1, \dots, a'_d}$ where

$$(19) \quad 0 < a'_i < a'_d = a_d.$$

Lastly, let $w \in \text{int}(T_{a_1, \dots, a_d})$ and let (λ_i) be the barycentric coordinates of w relative to T_{a_1, \dots, a_d} , where

$$w = \lambda_{d+1} \cdot \mathbf{0} + \lambda_d \cdot (a_1, \dots, a_d) + \sum_{i=1}^{d-1} \lambda_i e_i.$$

The d -th coordinate of w is $\lambda_d \cdot a_d$, and for $1 \leq i \leq d-1$, the i -th coordinate is

$$(20) \quad \lambda_i + \lambda_d \cdot a_i = \lceil \lambda_d \cdot a_i \rceil.$$

We can conclude that every point in $\text{int}(T_{a_1, \dots, a_d}) \cap \mathbb{Z}^d$ is completely determined by its d -th coordinate. Equivalently, no two interior lattice points can have the same d -th coordinate.

Theorem 19. *Let $d \geq 3$ and $k \geq 1$. If $T \in \mathcal{S}_k^d$, and $\text{Vol}(T) = \frac{1}{d!}(dk+1)$, then $T \simeq T_{a_1, \dots, a_d}$, where $(a_1, \dots, a_{d-1}, a_d) = (dk, \dots, dk, dk+1)$.*

Proof. Let $\mathcal{V}(T)$, v_{d+1} , \mathcal{W} , $S_{i,j}$, and \mathcal{T} be as in the proof of Corollary 17. For $S_{1,d} \in \mathcal{T}$, Lemma 18 implies there exists a unimodular transformation f such that $f(v_{d+1}) = \mathbf{0}$, $f(w_1) = (1, \dots, 1)$, and $f(v_j) = e_j$ for $1 \leq j \leq d-1$. Let

$$f(v_d) = (a_1, \dots, a_d).$$

We may assume $0 < a_j \leq a_d$ by (19). Since $\text{Vol}(T) = \frac{1}{d!}(dk+1)$ and unimodular transformations preserve volume, (17) implies $a_d = dk+1$. Unimodular transformations also preserve interior lattice points, so $f(w_1) \in \text{int}(f(T)) \cap \mathbb{Z}^d$, and $f(w_1)$ is collinear with $\mathbf{0}$. Corollary 16 implies $f(w_i) = (i, \dots, i)$. If $(\lambda_{i,j})$ are the barycentric coordinates of $f(w_i)$, where

$$f(w_i) = \lambda_{i,d+1} \cdot \mathbf{0} + \lambda_{i,d} \cdot (a_1, \dots, a_d) + \sum_{j=1}^{d-1} \lambda_{i,j} \cdot e_j,$$

then (20) implies

$$\lambda_{i,j} + \lambda_{i,d} \cdot a_j = i.$$

Note that $0 < \lambda_{i,j} < 1$, and

$$f(w_i) \in \text{int}(f(T)) \cap \mathbb{Z}^d \iff w_i \in \text{int}(T) \cap \mathbb{Z}^d \iff 1 \leq i \leq k.$$

Since

$$\lambda_{i,d} \cdot a_j = (i-1) + 1 - \lambda_{i,j} > (i-1)\lambda_{i,d} + 1 - \sum_{\substack{j=1 \\ j \neq d}}^{d+1} \lambda_{i,j} > i \cdot \lambda_{i,d},$$

$a_j > i$ for all i , which implies $a_j > k \geq 1$. The final step is to show that $a_j = dk$.

Consider the interior point $f(w_k) = (k, k, \dots, k)$ and let (λ_j) be the barycentric coordinates of $f(w_k)$. Clearly $\lambda_d = \frac{k}{dk+1}$. By (20), we have

$$\lambda_j = k - a_j \cdot \lambda_d = k - \frac{a_j \cdot k}{dk+1}.$$

Since $\sum_{j=1}^{d+1} \lambda_j = 1$, it follows that

$$0 < \sum_{j=1}^d \lambda_j = (d-1)k - \frac{k}{dk+1}(a_1 + \dots + a_{d-1} - 1) < 1.$$

Using elementary algebraic manipulation, we obtain

$$1 \leq (kd+1)(d-1) - (a_1 + \dots + a_{d-1} - 1) \leq d,$$

which implies

$$(21) \quad a_1 + \dots + a_{d-1} \geq kd(d-1).$$

Finally, $a_j < a_d = dk+1$ and (21) imply $a_j = dk$ for $1 \leq j \leq d$. ■

Corollary 20. *If $T \in \mathcal{S}_k^d$, then $\text{Vol}(T) = \frac{1}{d!}(dk+1)$ if and only if $T \simeq S_d(k)$.*

Proof. We first show that $\text{Vol}(S_d(k)) = \frac{1}{d!}(dk+1)$. Using determinants,

$$\text{Vol}(S_d(k)) = \frac{1}{d!} \cdot |\det(\mathbf{A})|,$$

where

$$\mathbf{A} = \begin{bmatrix} e_2 - e_1 \\ e_3 - e_1 \\ \vdots \\ e_{d-1} - e_1 \\ e_d - e_1 \\ (-k, -k, \dots, -k) - e_1 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & & & & \\ -1 & & & & \\ -1 & & & & \\ \hline -k-1 & -k & \dots & -k \end{bmatrix}}_{d \times d}.$$

Let \mathbf{A}' be the matrix obtained by adding k times the sum of the first $d-1$ rows of \mathbf{A} to row d of \mathbf{A} . That is,

$$\mathbf{A}' = \begin{bmatrix} -1 & & & & \\ -1 & & & & \\ -1 & & & & \\ \hline -dk-1 & 0 & \dots & 0 \end{bmatrix}.$$

It is easy to check that $|\det(\mathbf{A})| = |\det(\mathbf{A}')| = dk + 1$. If $T \simeq S_d(k)$, then $\text{Vol}(T) = \text{Vol}(S_d(k)) = \frac{1}{d!}(dk + 1)$. On the other hand, if $\text{Vol}(T) = \frac{1}{d!}(dk + 1)$, then $T \simeq S_d(k)$ by Theorem 19 provided $S_d(k)$ is clean. Since $\text{Vol}(S_d(k)) = \frac{1}{d!}(dk + 1)$ and any lattice triangulation of $\mathcal{V}(S_d(k)) \cup \text{int}(S_d(k)) \cap \mathbb{Z}^d$ contains at least $dk + 1$ simplices, (3) and Corollary 5 imply $S_d(k)$ cannot have any lattice points on its boundary except for its vertices. \blacksquare

Let $f : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ be the translation

$$f(v) = v \cdot \mathbf{I}_d + k \cdot \sum_{i=1}^d e_i = v + (k, k, \dots, k),$$

and let $g : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ be the transformation $g(v) = v \cdot \mathbf{M}$ where

$$\mathbf{M} = \underbrace{\begin{bmatrix} 1-k & -k & -k & \dots & -k & -k \\ -k & 1-k & -k & \dots & -k & -k \\ -k & -k & 1-k & \dots & -k & -k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -k & -k & -k & \dots & 1-k & -k \\ (d-1)k & (d-1)k & (d-1)k & \dots & (d-1)k & (d-1)k+1 \end{bmatrix}}_{d \times d}.$$

That is, if $m_{i,j}$ is the entry in the row i and column j of \mathbf{M} , then

$$m_{i,j} = \begin{cases} 1-k, & i = j \neq d, \\ (d-1)k, & j = d, i \neq d \\ (d-1)k+1, & i = j = d \\ -k, & \text{otherwise.} \end{cases}$$

Consider the image of $\mathcal{V}(S_d(k))$ under the composition $g \circ f$. For $v \in \mathcal{V}(S_d(k))$, the j -th coordinate of v under $g \circ f$ is simply the dot product of $v + (k, \dots, k)$ with the j -th column of \mathbf{M} . It is easy to check that

$$(g \circ f)(v) = \begin{cases} \mathbf{0}, & v = -k \sum_{i=1}^d e_i, \\ e_i, & v = e_i, 1 \leq i \leq d-1, \\ (a_1, \dots, a_d), & v = e_d. \end{cases}$$

The composition $g \circ f$ is in fact a unimodular transformation which maps $S_d(k)$ to T_{a_1, \dots, a_d} , where the a_i are as in Theorem 19.

8. COUNTEREXAMPLES IN \mathbb{R}^2

The collinearity property of the interior lattice points does not hold for all clean triangles in \mathbb{R}^2 . Pick's theorem states that if P is a lattice polygon, then

$$(22) \quad \text{Vol}(P) = k + \frac{b}{2} - 1$$

where $k = |\text{int}(P) \cap \mathbb{Z}^2|$ and $b = |\partial P \cap \mathbb{Z}^2|$. For clean triangles, (22) reduces to

$$\text{Vol}(P) = k + \frac{1}{2}.$$

Thus all clean lattice triangles satisfy the minimal volume condition. In [22], Reznick proved that any $T \in S_k^2$ is equivalent to some $T_{a, 2k+1}$, where $0 < a < 2k+1$. Using this representation for clean lattice triangles, Reznick then showed that the number of equivalence classes of clean k -point lattice triangles increases with k . Thus there exist clean lattice triangles whose interior points are not collinear.

Consider the triangle $\Delta_{p,q}$ in \mathbb{R}^2 with vertex set $\{(-1, 0), (0, q), (p, -1)\}$, where $1 \leq p \leq q$ and $\gcd(p, q+1) = 1$. It is easy to check that $\Delta_{p,q}$ is clean. We first compute the number of interior points of $\Delta_{p,q}$. Using determinants,

$$\text{Vol}(\Delta_{p,q}) = \frac{q}{2}(p+1) + \frac{1}{2}.$$

Pick's theorem implies

$$(23) \quad |\text{int}(\Delta_{p,q}) \cap \mathbb{Z}^2| = \frac{q}{2}(p+1).$$

For $p = 1$, the interior points of $\Delta_{p,q}$ are all on the line $x = 0$. However, for $p > 1$, this is not the case as they are covered by the lines $x = i$ ($0 \leq i \leq p-1$). Since these lines are parallel, they cannot be contained within any one line. Incidentally, this collection of counterexamples is related to the following summation identity.

Proposition 21. *If $(p, q) \in \mathbb{Z}_+^2$ and $\gcd(q+1, p) = 1$, then*

$$(24) \quad \sum_{i=0}^{p-1} \left\lceil q - \frac{i(q+1)}{p} \right\rceil = \frac{q}{2}(p+1).$$

Proof. This is a simple counting argument using (23), the fact that the lines $x = i$ ($0 \leq i \leq p-1$) cover $\text{int}(\Delta_{p,q}) \cap \mathbb{Z}^2$, and the fact that there are

$$\left\lceil q - \frac{i(q+1)}{p} \right\rceil$$

lattice points on the line $x = i$ and $\text{int}(\Delta_{p,q})$. ■

The counterexample above can be generalized to a triangle with vertex set

$$\{(-r, 0), (0, q), (p, -1)\},$$

where p , q , and r are positive integers and $\gcd(r, q) = \gcd(p, q+1) = 1$. The corresponding identity below is similar to (24).

Proposition 22. *If $(p, q, r) \in \mathbb{Z}_+^3$ and $\gcd(q, r) = \gcd(q + 1, p) = 1$, then*

$$\sum_{i=1}^{r-1} \left\lceil \frac{iq}{r} \right\rceil + \sum_{i=0}^{p-1} \left\lceil q - \frac{i(q+1)}{p} \right\rceil = \frac{q(p+r) + (r-1)}{2}.$$

Pick's theorem itself is a counterexample since only in \mathbb{R}^2 is the volume of a lattice polytope P an equality in terms of the number of lattice points on ∂P and in $\text{int}(P)$. Reeve's tetrahedra [20] have vertices

$$\{ (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, n) \},$$

where $n \in \mathbb{Z}_+$. These tetrahedra are clearly clean and empty, yet their volume increases with n . Reeve concluded that the volume of lattice of a polyhedron P cannot be expressed solely in terms of $b = |\partial P \cap \mathbb{Z}^3|$ and $k = |\text{int}(P) \cap \mathbb{Z}^3|$. However, Reeve was able to find an analogue to Pick's theorem by considering sublattices.

Let L_n^d denote the lattice consisting of all points in \mathbb{R}^d whose coordinates are multiples of $\frac{1}{n}$. Note that $L_1^d = \mathbb{Z}^d$. For a lattice d -polytope P , let

$$b_n = b_n(P) = |\partial P \cap L_n^d| \quad \text{and} \quad k_n = k_n(P) = |\text{int}(P) \cap L_n^d|.$$

Reeve showed in [21] that

$$(25) \quad 2n(n^2 - 1) \cdot \text{Vol}(P) = b_n - nb_1 + 2(k_n - nk_n) + (n - 1)[2\chi(P) - \chi(\partial P)],$$

where $n \geq 2$ and χ is the Euler characteristic. Reeve also conjectured a similar formula for $d = 4$. Soon after, Macdonald [16] not only confirmed Reeve's conjecture, but generalized Reeve's formulas to

$$(26) \quad (d - 1)d! \cdot \text{Vol}(P) = \sum_{i=1}^{d-1} (-1)^{i-1} \binom{d-1}{i-1} (b_{d-i} + 2k_{d-i}) \\ + (-1)^{d-1} [2\chi(P) - \chi(\partial P)].$$

In 1996, Kolodzieczyk [11] showed that (26) has the following alternate form in which only the numbers k_n of interior lattice points in the sublattice L_n are required.

$$(27) \quad d! \cdot \text{Vol}(P) = \sum_{i=0}^{d-1} (-1)^i \binom{d}{i} k_{d-i} + (-1)^d [\chi(P) - \chi(\partial P)].$$

For a d -polytope P all of whose vertices join d edges, $\chi(P) - \chi(\partial P) = (-1)^d$ and (27) assumes the form

$$(28) \quad d! \cdot \text{Vol}(P) = \sum_{i=0}^{d-1} (-1)^i \binom{d}{i} k_{d-i} + 1.$$

Recently, Kolodzieczyk showed in [12] that Pick-type formulas exist even when considering only the points in $L_n \cap P$ whose coordinates are odd multiples of $\frac{1}{n}$.

Though no formula for the volume of a lattice polytope in terms of the number of its boundary and interior lattice points exist, there does exist a Pick-type inequality with sharp bounds on the volume of a polyhedron. Any polyhedron in \mathbb{R}^3 that does not intersect itself has an associated planar graph. We can use elementary graph theory and Euler's formula to obtain a lower bound on the volume of lattice polyhedra.

Proposition 23. *Let $P \subset \mathbb{R}^3$ be a convex lattice polyhedron with b lattice points on the boundary and $k \geq 1$ interior lattice points. Then*

$$\text{Vol}(P) \geq \frac{2b + 3k - 7}{6}.$$

Proof. By (2), it suffices to show P has a lattice triangulation \mathcal{T} satisfying

$$|\mathcal{T}| \geq 2b + 3k - 7.$$

Any refinement sequence starting with any basic triangulation of P will suffice. Consider the graph G whose vertex set $\mathcal{V}(G)$ is $\partial P \cap \mathbb{Z}^3$ and whose edge set consists of the edges of ∂P . Since P is convex, G is planar. (Just embed G into \mathbb{S}^2 and then map onto \mathbb{R}^2 .) Thus every triangulation of G has $2 \cdot |\mathcal{V}(G)| - 4$ faces, implying ∂P can be triangulated into $2b - 4$ triangles. Since $k \geq 1$, P can be triangulated into $2b - 4$ subtetrahedra using any interior lattice point. The remaining interior points refine this triangulation, via Theorem 4, into a full triangulation having at least $2b - 4 + 3(k - 1)$ subtetrahedra. ■

This bound is sharp (consider any clean, non-empty lattice tetrahedron). A similar result is proved (independently) in [5, Lemma 3.5.5] by De Loera, Rambau, and Santos Leal. While their proof does not assume that P is nonempty, it does assume that $P \cap \mathbb{Z}^3$ is in general position. They also showed that any polyhedron with n vertices has a triangulation consisting of at most $2n - 7$ tetrahedra.

9. FUTURE RESEARCH

What can we say about an upper bound for the volume of polyhedra? Hensley proved in [9] that the volume of any lattice d -polytope is bounded above by a function in terms of d and the number of interior lattice points. Ziegler improved Hensley's results and showed in [14] that the volume of a k -point lattice d -polytope P satisfies

$$\text{Vol}(P) \leq k \cdot [7(k + 1)]^{d \cdot 2^{d+1}}.$$

In \mathbb{R}^3 , a few simple computations seem to indicate that the upper bound on the volume of clean tetrahedra is linear in k . Moreover, maximal-volume tetrahedra also appear to have collinear interior lattice points. Unlike the minimal-volume tetrahedra, however, the line containing the interior points does not pass through any vertex. Based on these computations, we make the following conjectures.

Conjecture 1. If $T \in \mathcal{S}_k^3$, then $\text{Vol}(T) \leq \frac{1}{6!}(12k + 8)$.

Conjecture 2. If $T \in \mathcal{S}_k^3$ and $\text{Vol}(T) = \frac{1}{6!}(12k + 8)$, then $T \simeq T_{2k+1, 4k+3, 12k+8}$. If $k > 1$, then $\text{int}(T) \cap \mathbb{Z}^3$ is a set of collinear points that does not pass through any vertex of T .

These conjectures are true for $k = 1$, as proved independently by Kasprzyk [10] and Reznick [23]. It may be possible to gain some new insight on the relationship between lattice points of a lattice d -polytope and its volume using Ehrhart theory. For example, it is well known that every convex lattice d -polytope is associated with an Ehrhart polynomial of degree d . Moreover, the leading coefficient of the associated polynomial is precisely the volume of the d -polytope (cf. [1], [6]).

10. ACKNOWLEDGEMENTS

I would like to thank my thesis advisor Bruce Reznick for having introduced me to lattice polytopes. I am also grateful to Phil Griffith, former director of graduate studies of the math department at the University of Illinois in Champaign-Urbana, who was directly responsible for my opportunity to work with Bruce Reznick. I also would like to thank Matthias Beck and Sinai Robins, the authors of [1], for their enlightening course on lattice point enumeration in Banff, Canada.

REFERENCES

- [1] M. Beck and S. Robins, *Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra*. Springer-Verlag, New York, NY, 2007.
- [2] C. Bey, M. Henk, and J. Wills, Notes on the Roots of Ehrhart Polynomials
<http://arxiv.org/abs/math.MG/0606089>
- [3] Marilyn Breen, Determining a Polytope by Radon Partitions. *Pacific Journal of Mathematics*, Vol. 43, No. 1, (1971) 27-37.
- [4] C. Carathéodory, Ueber den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen. *Rend. Circ. Mat. Palermo*, Vol. 32 (1911) 193-217.
- [5] J. A. De Loera, J. Rambau, and F. Santos Leal, *Triangulations: Applications, Structure, Algorithms*. (March 2007, to be published)
- [6] Eugène Ehrhart, Sur les polyèdres rationnels homothétiques à n dimensions. *C.-R. Acad. Sci.* Vol. 254 (1962) 616-618.
- [7] Branko Grünbaum, *Convex Polytopes*, 2nd ed. Springer-Verlag, New York, NY, 2003.
- [8] William R. Hare and John Kenelly, Characterizations of Radon Partitions. *Pacific Journal of Mathematics*, Vol. 36, No. 1, (1971) 159-164.
- [9] D. Hensley, Lattice vertex polytopes with interior lattice points. *Pacific Journal of Mathematics*, Vol. 105 (1983) 183-191.
- [10] A. Kasprzyk, Toric Fano 3-folds with terminal singularities. to appear in *Tokohu Math. J.*
- [11] K. Kolodziejczyk, A New Formula for the Volume of Lattice Polyhedra. *Monat. für Math.*, Vol. 122 (1996) 367-375.
- [12] ———, On Odd Points and the Volume of Lattice Polyhedra. *Journal of Geometry*, Vol. 68 (2000) 155-170.
- [13] L. Kosmak, A remark on Helly's theorem, *Spisy Přírod. Fak. Univ. Brno*, (1963) 223-225.
- [14] J. Lagarias and G. M. Ziegler, Bounds for lattice polytopes containing a fixed number of interior points in a sublattice, *Can. J. Math.*, Vol. 43 (1991), 1022-1035.
- [15] C. L. Lawson, Properites of n -dimensional triangulations. *Computer Aided Geometric Design* 3, 1986 (231-246).
- [16] I. G. Macdonald, The volume of a lattice polyhedron. *Proc. Cambridge Philos. Soc.*, Vol. 59 (1963) 719-726.
- [17] B. B. Peterson, The Geometry of Radon's Theorem. *The American Mathematical Monthly*, Vol. 79, No. 9, (Nov. 1972), 949-963.
- [18] I. V. Proskuryakov, A property of n -dimensional affine space connected with Helly's theorem. *Usp. Math. Nauk.*, Vol. 14, No. 1(85), (1959) 219-222.
- [19] J. Radon, Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten. *Math. An.*, Vol. 83 (1959), 219-222.
- [20] J. Reeve, On the volume of lattice polyhedra. *Proc. London Math. Soc. (3)*, Vol. 7 (1957), 378-395.
- [21] ———, A further note on the volume of lattice polyhedra. *J. London Math. Soc.*, Vol. 34 (1959) 57-62.
- [22] B. Reznick, Lattice Point Simplices, *Discrete Mathematics*, Vol. 60 (1986) 219-242.
- [23] ———, Clean Lattice Tetrahedra
<http://arxiv.org/abs/math.CO/0606227>
- [24] Günter M. Ziegler, *Lectures on Polytopes*. Springer-Verlag, New York, NY, 1995.